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Predictive Estimates in Population Models with Variable Dynamics Under Uncertainties

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Abstract

In this paper, we investigate the predictive estimates of the model which at the initial time interval describes a slower population growth, and later turns into a model with a rapid growth of such a population. For considered problem, with unknown initial conditions and parameters of differential equations, however for a known number of persons in the population at certain moments of time, we obtain the predictive sets at a given time under certain conditions and substantiate the formulas for calculating the minimum and maximum number of persons in the population.

KEY WORDS: *predictive estimate, uncertainty, population model, nonlinear equation, variable dynamics, information spreading.*

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1. Introduction

Mathematical models of population processes provide a framework for understanding the dynamics of populations over time. These models utilize mathematical equations to describe birth rates, mortality rates, immigration, and emigration, among other factors [1], [2]. By incorporating variables such as carrying capacity and environmental influences, these models can simulate the growth or decline of populations under various conditions, aiding in predicting future trends and informing decision-making in fields such as ecology, epidemiology, and economics.

In population processes, uncertainties manifest in various forms [3], including unknown parameters, which refer to incomplete knowledge about factors influencing population dynamics such as birth rates or mortality rates. Furthermore, error in observations introduces uncertainty [4], as inaccuracies or limitations in data collection methods can distort our perception of population trends, complicating efforts to accurately model and predict population behavior.

Population models with a fuzzy structure are also insufficiently studied. This type of uncertainty has been studied in works [5] – [7].

Mathematical models for describing population processes can take various forms, each tailored to capture specific aspects of population dynamics. Here are some different types of mathematical models commonly used:

Agent-Based Models: In agent-based models, populations are simulated as collections of individual agents, each with its own characteristics, behaviors, and interactions. These models are useful for capturing heterogeneity within populations and exploring emergent phenomena arising from individual-level interactions.

Spatial Models: These models incorporate spatial dimensions to analyze how population processes vary across geographical regions. They are valuable for studying patterns of migration, population distribution, and the spread of infectious diseases.

Stochastic Models: Stochastic models introduce randomness into population processes, acknowledging inherent variability and uncertainty. They are particularly useful for studying small populations, rare events, or systems subject to random fluctuations.

Nonlinear Differential Equations Models: Nonlinear differential equations are powerful tools for modeling population processes, allowing for a more realistic representation of complex dynamics such as feedback mechanisms, saturation effects, and interactions between different population groups.

We propose to use the model, which at the initial time interval describes a slower population growth, and later turns into a model with a rapid growth of such a population.

Initial Phase of Slower Population Growth: In the initial time interval, the population model may exhibit characteristics of slow growth. This phase could be influenced by factors such as limited resources, environmental constraints, or low reproductive rates. For example, a population may initially have a small number of individuals, undergoing gradual growth as individuals reproduce and the population expands to fill available niches. During this phase, the population growth rate may be moderate, and the population size increases steadily over time.

Transition to Rapid Growth: As the population approaches a critical threshold or experiences changes in its environment, it may undergo a transition to rapid growth. This transition could occur due to various factors, such as favorable environmental conditions, increased availability of resources, or changes in reproductive behaviors. For example, if a population's predators decline in number or if abundant food sources become available, individuals may experience reduced mortality rates or increased reproductive success, leading to accelerated population growth. This transition often involves nonlinear dynamics, where small changes in conditions can trigger disproportionately large responses in population growth rates.

Predictive estimates from a system of nonlinear differential equations are essential for understanding the dynamics of population growth, particularly when the model depicts transitions from slower to rapid growth phases. By providing insights into future trends and potential impacts [8], [9], predictive modeling informs policy, management, and planning efforts aimed at promoting sustainable development and addressing the challenges associated with population growth.

In this article, we explore modeling population processes with dynamics transitioning from slow to rapid using nonlinear differential equations to describe the processes. For each type of process behavior, a specific equation with parameters and initial conditions is proposed. We will also propose a method for obtaining predictive estimates of the dynamics of such models.

2. The Mathematical Background

Today, a significant number of publications are dedicated to the study of the behavior of dynamic systems. Such interest is caused by the variety of applications to real processes that are described using the mathematical apparatus of dynamic systems. The information spreading in social networks [10], the change in the number of patients during epidemics [11], the dynamics of the number of people with stress syndrome [1], [12] can be modeled with the help of hybrid dynamic systems. The most common models, which are studied, for example in [13] – [17], are logistic or more generalized Volterra models. With rapid population growth, it is recommended to use Gompertz models [18-22]. Given the known initial conditions and parameters of the models, such an analysis is reduced to solving Cauchy problems for linear or nonlinear differential equations.

In the case when the initial conditions or parameters of the models are unknown and belong to certain sets, it is not possible to obtain accurate predictive estimates, which forces us to look for the predictive sets at a given time under certain conditions. With unknown initial conditions and parameters of differential equations, however, for a known number of persons in the population at certain moments of time, formulas are given for calculating the minimum and maximum number of persons in the population.

3. Main Results

Let functions $x_1(t)$ and $x_2(t)$ be solutions of the system

$$\begin{aligned}
&\frac{dx_1}{dt} = \gamma_1(t)x_1(t)(N - x_1(t)), \quad x_1(0) = x_0, \quad 0 < t < t_1, \\
&x_1(t_1) = x_2(t_1), \\
&\frac{dx_2}{dt} = (b(t)u(t) + \gamma_2(t)\ln\frac{x_2(t)}{N})x_2(t), \quad t_1 < t < T.
\end{aligned} \tag{1}$$

Assume that the functions $\gamma_1(t), \gamma_2(t), u(t)$ are integrable with the square on the corresponding intervals, and a function $b(t)$ is continuous on $[t_1, T]$.

Proposition 1. *Equation system (1) has a unique solution that is continuous and differential almost everywhere (a. e.).*

Proof. Without limiting the generality, we will assume that $N = 1$.

Let's put $\varphi_1(t)=\dfrac{x_1(t)}{1-x_1(t)}=-1+(1-x_1(t))^{-1}$, $\varphi_2(t)=\ln x_2(t)$. Functions $\varphi_1(t)$ and $\varphi_2(t)$ are differential a.e. and almost everywhere satisfy the equations

$$
\begin{gathered}
\frac{d\varphi_1(t)}{dt}=\gamma_1(t)\varphi_1(t),\ \varphi_1(0)=-1+(1-x_1(0))^{-1} \\
at\ x_1(0)\neq 1\quad and\quad for\ \ 0<x_0<N, \\
\varphi_2(t_1)=\ln x_1(t_1)(1-x_1(t_1))^{-1}, \\
\frac{d\varphi_2}{dt}=b(t)u(t)+\gamma_2(t)\varphi_2(t).
\end{gathered}
\tag{2}
$$

We can write the solution of system (2) in the form

$$\varphi_1(t)=\varphi_1(0)\exp\int_0^t\gamma_1(\tau)d\tau, 0\le t\le t_1,$$

$$\varphi_2(t)=\varphi_2(t_1)\exp\int_{t_1}^t\gamma_2(\tau)d\tau+\int_{t_1}^t\exp\int_\tau^t\gamma_2(s)ds\cdot b(\tau)u(\tau)d\tau,$$

therefore, the solution of system (1) is functions $x_1(t)=\dfrac{\varphi_1(t)}{1+\varphi_1(t)}$, $x_2(t)=\exp\varphi_2(t)$. □

Further, let x_0 be an unknown value that belongs to interval $[x_0^-,x_0^+]$, where $x_0^->0$, $x_0^+<1$.

Let also $x_1(s_k)$ be the given values with errors $v_k, k=\overline{1,m}$ at the points $s_1,s_2,...,s_m$, $0<s_1<s_2<...<s_m<t_1$ with some value $x_0\in[x_0^-,x_0^+]$. Also assume that we observe some given values

$$y_k=x_1(s_k)+v_k, k=\overline{1,m},\ \text{ and }\ v_k^-\le v_k\le v_k^+.$$

Let us introduce the set $G_y=\left\{x_0: x_0^-\le x_0\le x_0^+, v_k^-\le y_k-x_1(s_k)\le v_k^+, k=\overline{1,m}\right\}$, and also the sets $G_y^{(1)}$ and $G_y^{(2)}$

$$G_y^{(1)}=\left\{x_1(\bar t,x_0): x_0\in G_y, s_m<\bar t\le t_1\right\},$$

$$G_y^{(2)}=\left\{x_2(t,x_1(t_1,x_0)): x_0\in G_y, t_1<\bar t<T\right\}.$$

The sets $G_y^{(1)}$ and $G_y^{(2)}$ are called the sets of predictive estimates for values $x_1(\bar t)$ and $x_2(\bar t)$, respectively.

Proposition 2. *The set G_y has the form*

$$G_y=[\alpha,\beta]\cap[x_0^-,x_0^+]=[\max(\alpha,x_0^-),\min(\beta,x_0^+)]=[\alpha^-,\alpha^+].$$

Proof. From the inequality $v_k^-\le y_k-x_1(s_k)\le v_k^+$, according to the relation $x_1(s_k)=\dfrac{\varphi_1(s_k)}{1+\varphi(s_k)}$, we obtain such inequalities

$$\frac{y_k^-}{1+y_k^-}\le\varphi_1(s_k)\le\frac{y_k^+}{1+y_k^+},$$

where $y_k^-=y_k-v_k^+$, $y_k^+=y_k-v_k^-$.

Since $\varphi_1(s_k)=x_0\exp\int_0^{s_k}\gamma_1(\tau)d\tau,$ then for x_0 the ratios $\gamma_k^-\le x_0\le\gamma_k^+, k=\overline{1,m}$ are valid, and here $\gamma_k^-=\dfrac{y_k^-}{1+y_k^-}\exp\left\{-\int_0^{s_k}\gamma_1(\tau)d\tau\right\}$, $\gamma_k^+=\dfrac{y_k^+}{1+y_k^+}\exp\left\{-\int_0^{s_k}\gamma_1(\tau)d\tau\right\}$.

In this way, we get that $x_0\in[\alpha,\beta]$, $\alpha=\max\limits_{1\le k\le m}\gamma_k^-, \beta=\min\limits_{1\le k\le m}\gamma_k^+$. □

Consequence 1. *The sets $G_y^{(1)}$ and $G_y^{(2)}$ have the form $G_y^{(1)}=[\delta_1^-,\delta_1^+]$, $G_y^{(2)}=[\delta_2^-,\delta_2^+]$, where* $\delta_1^-=x_1(\bar t,\alpha^-)$, $\delta_1^+=x_1(\bar t,\alpha^+)$, $\delta_2^-=x_2(\bar t,x_1(t_1,\alpha^-))$, $\delta_2^+=x_2(\bar t,x_1(t_1,\alpha^+))$.

Suppose that the function $\gamma_1(t)$ is unknown and belongs to the set

$$\Gamma = \left\{ \gamma_1 : \int_0^{t_1} q^2(t)(\gamma_1(t) - \bar{\gamma}_1(t))^2 dt \le 1 \right\}, \tag{3}$$

where $\bar{\gamma}_1(t)$ is a well-known function integrable with the square, the function $q^2(t)$ is continuous on $[0,t_1]$ and such that the following inequality holds: $q^2(t) \ge q^2 > 0$.

We put that $x_0 \in [x_0^-, x_0^+]$. Let us denote by

$$\Gamma_y = \left\{ (x_0,\gamma_1) : x_0^- \le x_0 \le x_0^+, \gamma_1 \in \Gamma, v_k^- \le y_k - x_1(s_k) \le v_k^+ \right\}.$$

Definition 1. An interval $[x_1^-(\bar{t}), x_1^+(\bar{t})]$ is called a *set of predictive estimates* for the values $x_1(\bar{t})$, where $x_1^-(\bar{t}) = \min_{(x_0,\gamma_1)\in\Gamma_y} x_1(\bar{t}, x_0, \gamma_1)$, $x_1^+(\bar{t}) = \max_{(x_0,\gamma_1)\in\Gamma_y} x_1(\bar{t}, x_0, \gamma_1)$, $x_1(\bar{t}, x_0, \gamma_1)$ is the solution of system (1) at the initial value x_0 and the function γ_1. Similarly, for the predictive interval $[x_2(\bar{t}, x_1^-(t_1)), x_2(\bar{t}, x_1^+(t_1))]$ is called a set of predictive estimates for the values $x_2(\bar{t})$.

Proposition 3. *The following equalities*

$$x_1^-(\bar{t}) = 1 - (1 + \varphi_1^-(\bar{t}))^{-1}, \quad x_1^+(\bar{t}) = 1 - (1 + \varphi_1^+(\bar{t}))^{-1}, \tag{4}$$

hold, where $\varphi_1^-(\bar{t}) = \exp L^-(x_0,\gamma_1)$, $\varphi_1^+(\bar{t}) = \exp L^+(x_0,\gamma_1)$, and $L^-(x_0,\gamma_1) = \min_{\Gamma_y} L(x_0,\gamma_1)$, $L^+(x_0,\gamma_1) = \max_{\Gamma_y} L(x_0,\gamma_1)$, $L(x_0,\gamma_1) = \ln\varphi(0) + \int_0^{\bar{t}} \gamma_1(\tau)d\tau.$

Proof. Since $x_1(\bar{t}) = \dfrac{\varphi_1(\bar{t})}{1+\varphi_1(\bar{t})} = 1 - (1+\varphi_1(\bar{t}))^{-1}$, where $\varphi_1(\bar{t}) = \exp L(x_0,\gamma_1)$, then this representation yields equalities (4). □

Remark 1. Since $x_0 = \dfrac{\varphi(0)}{1+\varphi(0)}$, then the set Γ_y can be written as

$$\bar{\Gamma}_y = \Big\{ (\psi_0,\gamma_1) : \ \psi_0^- \le \psi_0 \le \psi_0^+, \gamma_1 \in \Gamma,$$
$$\theta_k^- \le L_k(\psi_0,\gamma_1) \le \theta_k^+, \quad k = \overline{1,m} \Big\},$$

where $L_k(\psi_0,\gamma_1) = \psi_0 + \int_0^{t_k} \gamma_1(\tau)d\tau$, $\psi_0 = \ln\varphi(0)$, $\psi_0^- = \ln\dfrac{x_0^-}{1-x_0^-}$, $\psi_0^+ = \ln\dfrac{x_0^+}{1-x_0^+}$, $\theta_k^- = \ln\dfrac{y_k^-}{1+y_k^-}$, $\theta_k^+ = \ln\dfrac{y_k^+}{1+y_k^+}$.

Remark 2. In order to find predictive sets for the value $x(\bar{t})$, one needs to find the minimum and maximum value of a linear functional $L_1(\psi_0,\gamma_1) = \psi_0 + \int_0^{\bar{t}} \gamma_1(\tau)d\tau$ on the convex and closed set $\bar{\Gamma}_y$.

Proposition 4. *Let* x_0 *and* γ_1 *belong to the set* Γ_y*, the functions* $b(t), u(t)$ *and* $\gamma_2(t)$ *are given. Then the predictive set for the value* $x_2(\bar{t})$ *as* $t_1 < \bar{t} < T$*, has the form* $[x_2^-(\bar{t}), x_2^+(\bar{t})]$*, where* $x_2^-(\bar{t})$ *and* $x_2^+(\bar{t})$ *are found from the solution of system (1) at* $x_2(t_1) = x_1^-(t_1)$ *and* $x_2(t_1) = x_1^+(t_1)$*, respectively.*

The proof of this statement follows from the representation $x_2(\bar{t}) = \exp\varphi_2(\bar{t})$ and equality $x_2^-(\bar{t}) = \exp\varphi_2^-(\bar{t}), x_2^+(\bar{t}) = \exp\varphi_2^+(\bar{t})$, where

$$\varphi_2^{\mp}(\bar{t}) = x_1^{\mp}(t_1)\exp\int_{t_1}^{\bar{t}} \gamma_2(\tau)d\tau + \psi(\bar{t}),$$

$$\bar{\psi}(t) = \int_0^{\bar{t}} \exp\int_{\tau}^{t} \gamma_2(s)ds\, b(\tau)u(\tau)d\tau.$$

Remark 3. If the predictive sets for the values $x_1(\bar{t})$ and $x_2(\theta)$ are given in the form of intervals $[x_1^-, x_1^+]$ and $[x_2^-, x_2^+]$, then the guaranteed predictive estimates and guaranteed predictive errors are calculated by the formulas (see, for example, [21])

$$\hat{x}_i = \frac{1}{2}(x_i^+ + x_i^-), \quad \sigma_i = \frac{1}{2}(x_i^+ - x_i^-), \quad i = 1,2.$$

Definition 2. Let Γ_y^- and Γ_y^+ be such sets that such sets connected by embedding $\Gamma_y^- \subset \Gamma_y \subset \Gamma_y^+$. The predictive sets for $x_1(\bar{t})$ and $x_2(\Theta)$, corresponding to the sets Γ_y^- and Γ_y^+ are called the *lower* and *upper predictive sets*.

Suppose that

$$F(\psi_0, \gamma_1) = (\psi_0 - \bar{\psi})^2 \sigma_0^{-2} +$$
$$+\int_0^{t_1} (\gamma_1(t) - \bar{\gamma}_1(t))^2 q^2(t)dt + \sum_{k=1}^{m} (L_k(\psi_0, \gamma_1) - \bar{\theta}_k)^2 \sigma_k^{-2},$$

where $\bar{\psi} = \frac{1}{2}(\psi_0^+ + \psi_0^-), \sigma_0 = \frac{1}{2}(\psi_0^+ - \psi_0^-), \bar{\theta}_k = \frac{1}{2}(\theta_0^+ + \theta_0^-), \sigma_k = \frac{1}{2}(\theta_k^+ - \theta_k^-).$

Further, consider the sets $\Gamma(\beta_i) = \{(\psi_0, \gamma_1) : F(\psi_0, \gamma_1) \le \beta_i\}$, $i = 1,2.$

Let's choose β_1 and β_2 in such a way that $\Gamma(\beta_1) \subset \Gamma_y$ and $\Gamma(\beta_2) \supset \Gamma_y$. We need to find the lower and upper prediction sets in this case.

Let's introduce a notation $(\hat{\psi}_0, \hat{\gamma}) \in Arg \min\limits_{\psi_0, \gamma_1} F(\psi_0, \gamma_1).$ First, we show that the following statement holds.

Proposition 5. *This equality holds*

$$F(\psi_0, \gamma_1) = F(\hat{\psi}_0, \hat{\gamma}) + F_1(\psi_0 - \hat{\psi}, \gamma_1 - \hat{\gamma}),$$

where $F_1(\psi, \gamma) = \psi_0^2 \sigma_0^{-2} + \int_0^{t_1} \gamma^2(\tau) d\tau + \sum_{k=1}^{m} L_k^2(\psi_0, \gamma) \sigma_0^{-2}.$

Proof. Consider a function $g(\tau) = F(\hat{\psi}_0 + \tau(\psi_0 - \hat{\psi}_0), \hat{\gamma} + \tau(\gamma_1 - \hat{\gamma}))$ and expand such a function into a Taylor series at a point $\tau = 0.$ Then we obtain

$$g(\tau) = g(0) + \frac{1}{2} g''(0) \tau^2.$$

Note that since $(\hat{\psi}_0, \hat{\gamma})$ is the minimum of the function $F(\psi_0, \gamma),$ then $g'(0) = 0.$ Since

$$g(\tau) = (\tau \tilde{\psi}_0 - \bar{\psi})^2 \sigma_0^{-2} +$$
$$+\int_0^{\bar{t}} (\tau \tilde{\gamma}(s) - \bar{\gamma}_1(s))^2 q^2(s) ds + \sum_{k=1}^{m} (\tau L_k(\tilde{\psi}_0, \tilde{\gamma}) - \bar{\theta}_k)^2 \sigma_k^{-2},$$

where $\tilde{\psi}_0 = \psi_0 - \hat{\psi}_0, \tilde{\gamma}(s) = \gamma_1(s) - \hat{\gamma}(s),$ then

$$\frac{1}{2} g''(0) = \sigma_\psi^{-2} \tilde{\psi}_0^2 + \sum_{k=1}^{n} L_k^2(\tilde{\psi}_0, \tilde{\gamma}) \sigma_k^{-2} + \int_0^{t_1} \tilde{\gamma}^2(s) q^2(s) ds.$$

From here we obtain the necessary equality. □

Consequence 2. *We can write sets* $\Gamma(\beta_i)$ *in the form*

$$\Gamma(\beta_i) = \{(\psi_0, \gamma) : F_1(\psi_0 - \hat{\psi}_0, \gamma_1 - \hat{\gamma}) \le \beta_i - F(\hat{\psi}_0, \hat{\gamma})\}.$$

Proof. We obtain that at $\tau = 1$

$$g(1) = g(0) + \frac{1}{2} g''(0).$$

Since $g(1) = F(\psi_0, \gamma_1),$ from the fact that $\frac{1}{2} g''(0) = F_1(\psi_0 - \hat{\psi}_0, \gamma_1 - \hat{\gamma})$ we obtain the necessary equality. □

Lemma 1. *The following equalities hold*

$$\max_{\Gamma(\beta_i)} L(\psi_0, \gamma) = L(\hat{\psi}_0, \hat{\gamma}) + \delta_2 (\beta_i - F(\hat{\psi}_0, \hat{\gamma}))^{\frac{1}{2}},$$
$$\min_{\Gamma(\beta_i)} L(\psi_0, \gamma) = L(\hat{\psi}_0, \hat{\gamma}) + \delta_1 (\beta_i - F(\hat{\psi}_0, \hat{\gamma}))^{\frac{1}{2}},$$
$$i = 1,2, \quad \delta_1 = \min_{\Gamma_0} L(\psi_0, \gamma), \quad \delta_2 = \max_{\Gamma_0} L(\psi_0, \gamma),$$
$$\Gamma_0 = \{(\psi_0, \gamma) : F_1(\psi_0, \gamma) \le 1\}.$$

Proof. Obviously, if we make a substitution $\psi_0 - \hat{\psi}_0 = \tilde{\psi}_0,\ \gamma_1 - \hat{\gamma} = \tilde{\gamma},$ we get the relation

$$\max_{\Gamma(\beta_i)} L(\psi_0, \gamma) = \max_{\Gamma_1(\beta_i)} L(\psi_0, \gamma) + L(\hat{\psi}_0, \hat{\gamma}) =$$

$$= (\beta_i - F(\hat{\psi}_0,\hat{\gamma}))^{\frac{1}{2}} \max_{\Gamma_0} L(\psi_0,\gamma) + L(\hat{\psi}_0,\hat{\gamma}),$$

where $\Gamma_1(\beta_i) = \{(\psi_0,\gamma) : F_1(\psi_0,\gamma) \le \beta_i - F(\hat{\psi}_0,\hat{\gamma})\}$.

We obtain similar relations for $\min_{\Gamma(\beta_i)} L(\psi_0,\gamma).$

Thus, in order to find predictive estimates, it is necessary to find values $\hat{\psi}_0$ and $\hat{\gamma}$, as well as expressions for δ_1 and δ_2.

Since $(\hat{\psi}_0,\tilde{\gamma}) \in Arg\min F(\psi_0,\gamma_1)$, then these values can be found from the equation

$$\frac{d}{d\tau} F(\hat{\psi}_0 + \tau v_0, \hat{\gamma} + \tau v_1)\Big|_{\tau=0} \equiv 0$$

for arbitrary numbers v_0 and functions $v_1(t)$ integrable with the square on $(0,t_1)$. □

Lemma 2. The following equality is valid

$$\frac{1}{2}\frac{d}{d\tau} F(\hat{\psi}_0 + \tau v_0, \hat{\gamma} + \tau v_1)\Big|_{\tau=0} = (\hat{\psi}_0 - \overline{\psi})\sigma_0^{-2} v_0 +$$
$$+\int_0^{t_1} (\hat{\gamma}(s) - \overline{\gamma}(s)) v_1(s) ds +$$
$$+\sum_{k=1}^{m} (L_k(\hat{\psi}_0,\hat{\gamma}) - \overline{\theta}_k)\sigma_k^{-2} L_k(v_0,v_1),$$

where $L_k(v_0,v_1) = v_0 + \int_0^{t_1} \chi_{[0,t_k]}(s) v_1(s) ds$, χ is the characteristic function of the interval $[0,t_k]$.

The proof of this Lemma 2 follows from the form of the functional $F(\psi_0,\gamma_1)$.

Proposition 6. *A pair of values $\hat{\psi}_0$ and $\hat{\gamma}(s)$ is a unique solution of a system*

$$\begin{cases} (\hat{\psi}_0 - \overline{\psi})\sigma_0^{-2} + \sum_{k=1}^{m} (L_k(\hat{\psi}_0,\hat{\gamma}) - \overline{\theta}_k)\sigma_k^{-2} = 0, \\ (\hat{\gamma}(s) - \overline{\gamma}(s)) + \sum_{k=1}^{m} (L_k(\hat{\psi}_0,\hat{\gamma}) - \overline{\theta}_k)\sigma_k^{-2} \chi_{[0,t_k]}(s) = 0. \end{cases}$$

Proof. We get these equations if we take into account the expression for the derivative of the function $F(\psi_0,\gamma)$ obtained in the Lemma 2, as well as the arbitrariness of the number v_0 and the function $v_1(s)$.

The uniqueness of the solution of these equations follows from the fact that the quadratic functional $F(\psi_0,\gamma)$ reaches a minimum at a unique point. □

Consequence 3. *The following equality is valid*

$$\hat{\psi}_0 = \left(\sigma_0^{-2} + m\sum_{k=1}^{m}\sigma_k^{-2}\right)^{-1} \left(\sum_{k=1}^{m}\overline{\theta}_k\sigma_k^{-2} - \sum_{k=1}^{m}\sigma_k^{-2} x_k\right),$$

$$\hat{\gamma}(s) = \overline{\gamma}(s) - \sum_{k=1}^{m}(\hat{\psi}_0 + x_k)\sigma_k^{-1}\chi_{[0,t_k]}(s) + \sum_{k=1}^{m}\overline{\theta}_k\sigma_k^{-2}\chi_{[0,t_k]}(s),$$

where numbers $x_k, k = \overline{1,m}$, can be found from the system of equations

$$x_j + \sum_{k=1}^{m} x_k \min(t_k,t_j)\sigma_k^{-2} =$$
$$= \sum_{k=1}^{m}\sigma_k^{-2}\overline{\theta}_k \min(t_k,t_j) - \sum_{k=1}^{m}\min(t_k,t_j)\hat{\psi}_o, \ j = \overline{1,m}.$$

Proof. The system of linear algebraic equations with respect to variables x_k can be obtained if we put $x_k = \int_0^{t_k} \hat{\gamma}(s) ds.$

Note that in order to find the values δ_1 and δ_2 it is necessary to calculate $\min L(\psi_0,\gamma)$ and $\max L(\psi_0,\gamma)$ on the set $\Gamma_0 = \{(\psi_0,\gamma) : F_1(\psi_0,\gamma) \le 1\}$.

Since the minimum and maximum of these expressions are reached on the boundary of the set Γ_0, there exist Lagrange multipliers λ_1 and λ_2 such that

$$\min L(\psi_0,\gamma) = L(\hat{\psi}(\lambda_1),\hat{\gamma}(\lambda_1)),$$

$$\max L(\psi_0,\gamma)=L(\hat{\psi}_0(\lambda_2),\hat{\gamma}(\lambda_2)),$$

where $\hat{\psi}_0(\lambda_i),\hat{\gamma}(\lambda_i)$ are the extremum points of the function $\mathcal{L}(\psi_0,\gamma)=L(\psi_0,\gamma)+\lambda F_1(\psi_0,\gamma)$ which can be found from the condition

$$\left.\frac{dg_i(t)}{dt}\right|_{t=0}\equiv 0.$$

$\forall v_0,v_1\quad g_1(t)=\mathcal{L}(\hat{\psi}_0(\lambda)+tv_0,\hat{\gamma}(\lambda)+tv_1),$ and Lagrange multipliers λ_1,λ_2 can be found from the equation $F_1(\hat{\psi}_0(\lambda),\hat{\gamma}(\lambda))=1.$ □

Let $u(t)=0,$ $\gamma_2(t)$ be an unknown function with the form

$$\gamma_2(t)=(\theta,g(t))=\sum_{k=1}^{m}\theta_k g_k(t),$$

where $g_k(t)$ are known piecewise continuous functions, $\theta=(\theta_1,...,\theta_m)^T$ is a vector of unknown parameters. Suppose that the values $y_k=x_2(\tau_k)+v_k,\ k=\overline{1,N},\ t_1<\tau_k<T$ are given, v_k are unknown values which belong to the interval $I_k=[v_k^-,v_k^+]$.

Definition 3. The set

$$G(\theta)=\{\theta: v_k^-\le y_k-x_2(\tau_k)\le v_k^+,k=\overline{1,N}\}$$

is called the *posterior set* of parameters θ.

Introduce the values $y_k^-=\ln\ln\dfrac{y_k-v_k^+}{\overline{\varphi}_2(t_1)},$ $y_k^+=\ln\ln\dfrac{y_k-v_k^-}{\varphi_2(t_1)},$ $g_k=\int_{t_1}^{\tau_k}g(\tau)d\tau,$ $g_0=\int_{t_1}^{T}g(\tau)d\tau.$ Let θ^+ and θ^- denote the solutions of linear programming problems

$$\max_{\theta\in\overline{G}(\theta)}(\theta,g_0)=(\theta^+,g_0),\ \min_{\theta\in\overline{G}(\theta)}(\theta,g_0)=(\theta^-,g_0),$$

where $\overline{G}(\theta)=\{\theta: y_k^-\le(\theta,g_k)\le y_k^+,\ k=\overline{1,N}\}$.

Definition 4. The expressions

$$\hat{x}_2(T)=\frac{x_2^+(T)+x_2^-(T)}{2},\quad \sigma_g=\frac{x_2^+(T)-x_2^-(T)}{2},$$

are called the *guaranteed predictive estimate* (GPE) of the value $x_2(T)$ and the *guaranteed predictive error* (GPEr) respectively, where $x_2^+(T)=\max\limits_{\theta\in G(\theta)}x_2(T),\quad x_2^-(T)=\min\limits_{\theta\in G(\theta)}x_2(T).$

We show that the following statement holds.

Theorem. Assume that the values $y_k,k=\overline{1,N}$ are given with their errors v_k, which belong to the interval I_k. Then equalities hold

$$\hat{x}_2(T)=\frac{1}{2}[\exp\varphi_+(T)+\exp\varphi_-(T)],$$

$$\sigma_g=\frac{1}{2}[\exp\varphi_+(t)-\exp\varphi_-(T)],$$

where $\varphi_+(T)=\varphi(t_1)\exp\{(\hat{\theta},g_0)+\sigma(\theta)\},$ $\varphi_-(T)=\varphi(t_1)\exp\{(\hat{\theta},g_0)-\sigma(\theta)\},$ $\hat{\theta}=\frac{1}{2}(\theta^++\theta^-),$ $\sigma(\theta)=\frac{1}{2}[(\theta^+,g_0)-(\theta^-,g_0)].$

Proof. Note that inequalities $v_k^-\le y_k-x_2(\tau_k)\le v_k^+$ can be written as $y_k-v_k^+\le x_2(\tau_k)\le y_k-v_k^-$. Since $x_2(\tau_k)=\exp\varphi_2(\tau_k)$, then for $\varphi(\tau_k)$ we obtain the inequality

$$\ln(y_k-v_k^+)\le\varphi_2(\tau_k)\le\ln(y_k-v_k^-).$$

Taking into account that $\varphi_2(\tau_k)=\varphi_2(t_1)\exp(\theta,g_k)$ we get that (θ,g_k) satisfies the inequalities $y_k^-\le(\theta,g_k)\le y_k^+$.

Moreover, $$\max_{\theta\in G(\theta)}x_2(T)=\max_{\theta\in G(\theta)}x_2(T)=\exp\max_{\theta\in\overline{G}(\theta)}\varphi_2(T)=\exp\varphi_2(t_1)\exp\max_{\theta\in\overline{G}(\theta)}(\theta,g_0)$$ and

$$\max_{\theta\in\overline{G}(\theta)}(\theta,g_0)=\frac{1}{2}\left((\theta^+,g_0)+(\theta^-,g_0)\right)+\frac{1}{2}\left((\theta^+,g_0)-(\theta^-,g_0)\right)=(\hat{\theta},g_0)+\sigma(\theta).$$

Similarly, we obtain the expressions $\min\limits_{\theta\in\overline{G}(\theta)}(\theta,g_0)=(\hat{\theta},g_0)-\sigma(\theta)$, which means $\min\limits_{\theta\in G(\theta)}x_2(T)=\exp\varphi_-(T).$ Taking these equalities into account, we obtain expressions for $\hat{x}_2(T)$ and σ_g. □

Remark 4. It follows from Theorem that in order to find the GPE $x_2(T)$ and the GPEr $x_2(T)$, it is necessary to find the GPE and the GPEr of the scalar product (θ, g_0) under the condition that the parameter θ belongs to the set $G(\theta)$.
Next, we find approximate the GPE and the GPEr for the value (θ, g_k).

We approximate the set $\bar{G}(\theta)$ by a set $G^-(\theta) = \left\{\theta : \sum_{k=1}^{N} (\hat{y}_k - (\theta, g_k))^2 q_k^{-2} \le 1\right\}$, where $\hat{y}_k = \frac{1}{2}(y_k^+ + y_k^-)$, $q_k = \frac{1}{2}(y_k^+ - y_k^-)$.

Lemma 3. *The following embedding holds* $G^-(\theta) \subset \bar{G}(\theta)$.

Proof. The inequalities $y_k^- \le (\theta, g_k) \le y_k^+$, $k = \overline{1, N}$, can be written in the form $\left|\hat{y}_k - (\theta, g_k)\right| \le q_k$, $k = \overline{1, N}$. From the condition $\sum_{k=1}^{N} \left|\hat{y}_k - (\theta, g_k)\right|^2 q_k^{-2} \le 1$ it follows the condition $|\hat{y}_k - (\theta, g_k)| \le q_k$, which means that $G^-(\theta) \subset \bar{G}(\theta)$. □

Further, let us introduce a matrix $P = \sum_{k=1}^{N} q_k^{-2} g_k g_k^T$. Let us denote by $\hat{\theta}$ the solution of the system of linear algebraic equations $P\hat{\theta} = \sum_{k=1}^{N} q_k^{-2} g_k \hat{y}_k$. Assume that $\det P \ne 0$. Then we show that the following statement holds.

Proposition 7. Approximate guaranteed posterior estimate of the expression (θ, g_0) has the form $(\hat{\theta}, g_0)$. At the same time, the approximate guaranteed posterior error of such an estimate can be written in the form

$$\sigma_H = (P^{-1} g_0, g_0)^{\frac{1}{2}} (1 - F(\hat{\theta}))^{\frac{1}{2}},$$

$$F(\theta) = \sum_{k=1}^{N} (\hat{y}_k - (\theta, g_k))^2 q_k^{-2}.$$

where

Proof. Let $\hat{\theta}$ denote the minimum point of the function $F(\theta)$. From the condition $F'(\hat{\theta}) \equiv 0$ we obtain that $\hat{\theta}$ satisfies the equation $P\hat{\theta} = \sum_{k=1}^{N} q_k^{-2} g_k \hat{y}_k$. Note that the set $\bar{G}(\theta)$ can be written as $\bar{G}(\theta) = \{\theta : F(\theta) \le 1\}$. From the expansion in the Taylor series at the point $\hat{\theta}$ we obtain that

$$F(\theta) = F(\hat{\theta}) + \left(P(\theta - \hat{\theta}), (\theta - \hat{\theta})\right).$$

From here we get

$$\frac{1}{2}\left(\max_{\theta \in \bar{G}(\theta)} (\theta, g) + \min_{\theta \in \bar{G}(\theta)} (\theta, g)\right) = (\hat{\theta}, g),$$

$$\sigma = \frac{1}{2}\left(\max_{\theta \in \bar{G}(\theta)} (\theta, g) - \min_{\theta \in \bar{G}(\theta)} (\theta, g)\right) =$$

$$= \max_{(P\theta, \theta) \le 1} (\theta, g)\left(1 - F(\hat{\theta})\right)^{\frac{1}{2}} = (P^{-1} g, g)\left(1 - F(\hat{\theta})\right)^{\frac{1}{2}},$$

which had to be proved. □

Remark 5. We can get a similar approximate estimate of the scalar product (θ, g_0) and its error when we approximate a set $\bar{G}(\theta)$ by a set

$$G^+(\theta) = \left\{\theta : \sum_{k=1}^{N} (\bar{y}_k - (\theta, g_k))^2 q_k^{-2} \le N\right\}.$$

In this case $\bar{G}(\theta) \subset G^+(\theta)$, and for the approximate guaranteed posterior error we obtain the expression

$$\sigma_H^+ = \left(P^{-1} g_0, g_0\right)^{\frac{1}{2}} \left(N - F(\hat{\theta})\right)^{\frac{1}{2}}.$$

Remark 6. In the case, when the approximate estimates of the scalar product (θ, g_0) and the approximate estimates of errors $\sigma(\theta)$ are given, then the approximate predictive estimate $x_2(T)$ is given in the form

$$\hat{x}_2^{(H)} = \frac{1}{2}\left(\exp \varphi_+^{(H)}(T) + \exp \varphi_-^{(H)}(T)\right),$$

where $\varphi_+^{(H)}(T) = \varphi(t_1) \exp\{(\theta, g_0)_H + \sigma_H(\theta)\}$, $\varphi_-^{(H)}(T) = \varphi(t_1) \exp\{(\theta, g_0)_H - \sigma_H(\theta)\}$, $(\theta, g_0)_H$ is the approximate estimate to (θ, g_0), $\sigma_H(\theta)$ is the approximate error of such an estimate.

Let us further consider the case when $u(t) \ne 0$, and $\gamma(t)$ is a known function.

Let's choose a function $u(t)$ from the condition $y_k = \varphi(s_k), k = \overline{1,N}$, $s_1 < s_2 < \ldots < s_N$, $s_k \in (t_1, T)$, y_k are given numbers.

Proposition 8. The set U for functions, for which the condition $y_k = \varphi(s_k), k = \overline{1,N}$, holds with given numbers y_k, has the form
$U = \{u : u(t) = u_0(t) + v(t)\}$,
where $u_0(T) = \sum_{k=1}^{N} x_k \Phi_k(t)$, x_k can be found from the system of linear algebraic equations $\sum_{j=1}^{N} b_{kj} x_j = c_k, k = \overline{1,N}$, and $v(t)$ is an arbitrary function from space $L_2(t_1, s_N)$ that satisfies the condition

$$\int_{t_1}^{s_N} \Phi_k(t) v(t) dt = 0, k = \overline{1,N},$$

where $\Phi_k(t) = \exp \int_t^{s_k} \gamma(\tau) d\tau$, $b_{kj} = \int_{t_1}^{s_N} \Phi_k(t)\Phi_j(t)\chi_k(t)\chi_j(t)\, dt$, $c_k = \bar{y}_k - \varphi(t_1)\Phi_k(t_1)$, $\bar{y}_k = \ln y_k$, $\chi_k(t) = \begin{cases} 1, t \in (t_1, s_k) \\ 0, t \notin (t_1, s_k) \end{cases}$.

Proof. Let us rewrite the condition $\varphi(s_k) = y_k$ in the form $\int_{t_1}^{s_k} \Phi_k(t) u(t) dt = c_k, k = \overline{1,N}$. The solutions of such a system have the form $u(t) = u_0(t) + v(t)$, where $u_0(t) = \sum_{j=1}^{N} x_j \Phi_j(t) \chi_j(t)$, which had to be proved. □

4. Numerical Experiment

Following the form of predictive sets, established by formula (3), the proposed algorithm is tested on synthetic data. We assume that the parameters of the model are stationary. We observe the state x_1 of the first equation of system (1) on the interval $t \in [0,4]$, then on the interval $t \in [4,5]$ we plot the dynamics of the first equation of system (1), which we no longer observe, then on the interval $t \in [5,10]$ we predict x_2 according to the specified parameters of our system (1). The experiment was conducted using Python's libraries *pandas, numpy, math* and *matplotlib.pyplot*.

In formula (3) we put
$\gamma_1 = 0.1$,
$\bar{\gamma}_1 = 0.25$,
$q = 10$,
$\gamma_2 = 0.3$
$u = 0.3$,
$b = 0.5$,
$x_0^- = 0.001$,
$x_0^+ = 0.02$.

Let's find guaranteed estimates for the above parameters

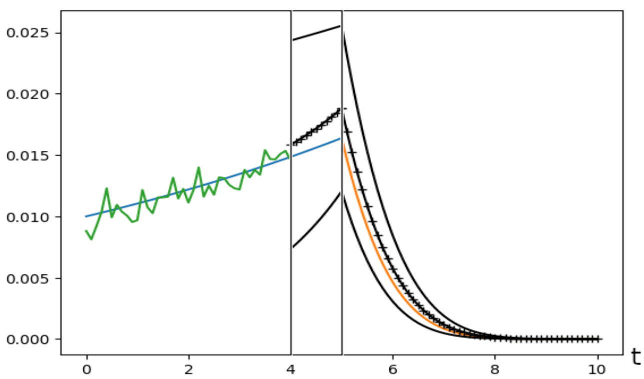

Fig. 1. Graph of system (1) behavior with the given parameters.

On Fig. 1 $x_1(t)$ is in blue, $x_2(t)$ is in orange, observation of $x_1(t)$ is in green. The black dashed curve shows the estimation error, the other black curve shows the predicted estimates of x_1 and x_2.

5. Conclusions

The research provides formulas for calculating predictive estimates of the number of individuals in the population with unknown non-stationary parameters included in the right-hand sides of special nonlinear differential equations. The obtained results can be applied in the tasks of analyzing the dynamics of the number of persons who received certain information, the dynamics of the number of persons with stress syndrome, the dynamics of the number of sick persons during epidemics.

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